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RATIONALITY OF SEMIALGEBRAIC FUNCTIONS

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ABSTRACT. Let X be an algebraic subset of \mathbb{R}^n , and $f: X \rightarrow \mathbb{R}$ a semialgebraic function. We prove that if f is continuous rational on each curve $C \subset X$ then: 1) f is arc-analytic, 2) f is continuous rational on X . As a consequence we obtain a characterization of hereditarily rational functions recently studied by J. Kollár and K. Nowak.

1. INTRODUCTION

Our goal is to give a short introduction to some results on real rational functions. The interested reader may consult [4] and [5] for a more comprehensive treatment. We strive for simplicity of our presentation. Keeping this in mind, we deal only with semialgebraic functions. Furthermore, we explain in detail the role of Bertini's theorem in establishing a criterion for rationality of such functions.

Throughout this section, X denotes an algebraic subset of \mathbb{R}^n . Recall that a function $f: X \rightarrow \mathbb{R}$ is said to be *regular* at $x \in X$ if there exist two polynomials $p, q \in \mathbb{R}[x_1, \dots, x_n]$ such that $q(x) \neq 0$ and $f = p/q$ in a Zariski open neighborhood of x in X . Furthermore, f is regular on a subset of X if it is regular at each point of this subset. We say that the function $f: X \rightarrow \mathbb{R}$ is *rational* if it is regular on a Zariski open dense subset of X (this minor deviation from the standard definition is justified, f being defined everywhere on X). Obviously, f is rational if and only if its restriction to each irreducible component of X is rational. We also say that the function f is *continuous rational* if it is continuous (in the strong topology) on X and rational in the sense just defined.

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Example 1.1. Let $C = \{x^3 = y^2\} \subset \mathbb{R}^2$, and let $f: C \rightarrow \mathbb{R}$ be the function defined by $f = y/x$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Then f is continuous rational on C but is not regular at $(0, 0)$.

Note that Example 1.1 makes sense also in the complex setting.

Example 1.2. The function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$, defined by $f(x, y) = \frac{x^3}{x^2+y^2}$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$, is continuous rational on \mathbb{R}^2 but is not regular at $(0, 0)$.

Example 1.2 is specific to real algebraic geometry. Indeed, in the complex setting, a continuous rational function on a nonsingular algebraic set is actually regular.

Continuous rational functions on nonsingular real algebraic sets are studied by the first named author [6, 7, 8] in the context of approximation of continuous maps into spheres. Both authors initiated a theory of vector bundles [9] on real algebraic varieties, in which continuous rational functions are used to define morphisms. Continuous rational functions, under the name *fonctions régulières*, are the object of investigation in [3]. An interesting phenomenon discovered by J. Kollár is recalled below.

Example 1.3. The algebraic surface

$$S := \{x^3 - (1 + z^2)y^3 = 0\} \subset \mathbb{R}^3$$

is an analytic submanifold, and the function $f: S \rightarrow \mathbb{R}$ defined by $f(x, y, z) = (1 + z^2)^{1/3}$ is analytic and semialgebraic. Furthermore, f is continuous rational on S since $f(x, y, z) = x/y$ on S without the z -axis. On the other hand, f restricted to the z -axis is not a rational function, and f cannot be extended to a continuous rational function on \mathbb{R}^3 , cf. [5].

In order to avoid such a pathology Kollár and Nowak [5] proposed the following notion. A function $f: X \rightarrow \mathbb{R}$ is said to be *hereditarily rational* if for every algebraic set $Z \subset X$ the restriction $f|_Z$ is a rational function on Z . According to the main result of [5], any continuous and hereditarily rational function on $X \subset \mathbb{R}^n$ can be extended to a continuous and hereditarily rational function on \mathbb{R}^n . Moreover, if the algebraic set X is nonsingular, then any continuous rational function on X is hereditarily rational [5].

Now we introduce a crucial notion for this paper. We say that a function $f: X \rightarrow \mathbb{R}$ is *curve-rational* if for every irreducible algebraic curve $C \subset X$ the restriction $f|_C$ is rational on C .

Our main result, whose proof is given in Section 3, is the following.

Theorem A. *For a function $f: X \rightarrow \mathbb{R}$ on an algebraic subset X of \mathbb{R}^n , the following conditions are equivalent:*

- (a) *f is hereditarily rational.*
- (b) *f can be extended to a hereditarily rational function on \mathbb{R}^n .*
- (c) *f is semialgebraic and curve-rational.*

As demonstrated by Example 1.2, a semialgebraic continuous rational function need not be curve-rational.

We are now heading towards our second result, which is proved in Section 4. We say that a function $f: X \rightarrow \mathbb{R}$ is *continuously curve-rational* if for every irreducible algebraic curve $C \subset X$ the restriction $f|_C$ is continuous rational on C .

In the 1980's the notion of arc-analytic function was introduced by the second named author [10]. A function $f: V \rightarrow \mathbb{R}$, defined on a real analytic set V , is said to be *arc-analytic* if $f \circ \gamma$ is analytic for every analytic arc $\gamma: (-1, 1) \rightarrow V$. An arc-analytic function on \mathbb{R}^n need not be continuous [1] and may have a nondiscrete singular set [11].

Theorem B. *Any semialgebraic, continuously curve-rational function on an algebraic subset of \mathbb{R}^n is continuous and arc-analytic.*

As an immediate consequence of Theorems A and B we obtain the following characterization of continuous hereditarily rational functions.

Corollary 1.4. *A function on an algebraic subset of \mathbb{R}^n is continuous and hereditarily rational if and only if it is semialgebraic and continuously curve-rational.*

In Section 2, which is crucial for the proof of Theorem A, we recall some results on semialgebraic sets, prove a suitable version of Bertini's theorem, and analyze the Zariski closure of a Nash submanifold of \mathbb{R}^n .

2. PRELIMINARIES

We will use the notion of *dimension* in various situations. If Y is an algebraic subset of \mathbb{R}^n or \mathbb{C}^n , then by $\dim Y$ we mean the Krull dimension of the ring of polynomial \mathbb{R} -valued or \mathbb{C} -valued, respectively, functions on Y . Recall that a Nash submanifold of \mathbb{R}^n is an analytic submanifold that is also a semialgebraic set. For a semialgebraic subset A of \mathbb{R}^n ,

$$\dim A := \max \dim N,$$

where N runs through the collection of Nash submanifolds of \mathbb{R}^n contained in A . The Zariski closure of an arbitrary subset S of \mathbb{R}^n will be denoted by $\overline{S}^{\mathcal{Z}}$. Hence $\overline{S}^{\mathcal{Z}}$ is the smallest algebraic subset of \mathbb{R}^n containing S . If X is an algebraic subset of \mathbb{R}^n , we denote by $X_{\mathbb{C}}$ its *complexification*, that is, the smallest algebraic subset of \mathbb{C}^n containing X . For A as above,

$$\dim A = \dim \overline{A}^{\mathcal{Z}} = \dim (\overline{A}^{\mathcal{Z}})_{\mathbb{C}},$$

cf. [2, Section 2.8]. We will frequently use these equalities without explicitly referring to them.

In the sequel we will also use the following standard facts on rational functions. Let X be an irreducible algebraic subset of \mathbb{R}^n . Recall that each nonempty Zariski open subset of X is Zariski dense. Let $f: X \rightarrow \mathbb{R}$ be a rational function that is

regular in a nonempty Zariski open set $X_0 \subset X$. We denote by $G(f)$ the Zariski closure in $\mathbb{R}^n \times \mathbb{R}$ of the graph of $f|_{X_0}$, that is,

$$G(f) := \overline{\text{graph}(f|_{X_0})}^{\mathcal{Z}}.$$

Since X is irreducible, so are X_0 and $\text{graph}(f|_{X_0})$. Consequently, $G(f)$ is irreducible as well. We have

$$\dim G(f) = \dim(\text{graph}(f|_{X_0})) = \dim X_0 = \dim X.$$

It readily follows that $G(f)$ does not depend on the choice of X_0 . By complexifying $X \subset \mathbb{R}^n$ and $G(f) \subset \mathbb{R}^n \times \mathbb{R}$, we obtain irreducible complex algebraic sets $X_{\mathbb{C}} \subset \mathbb{C}^n$ and $G(f)_{\mathbb{C}} \subset \mathbb{C}^n \times \mathbb{C}$. For our purposes, the key property of $G(f)_{\mathbb{C}}$ is the following.

Lemma 2.1. *Let $\pi: \mathbb{C}^n \times \mathbb{C} \rightarrow \mathbb{C}^n$ be the canonical projection. With notation as above,*

$$G(f)_{\mathbb{C}} \cap \pi^{-1}(x) = \{(x, f(x))\}$$

for all $x \in X_0$.

Proof. We can choose polynomials $p, q \in \mathbb{R}[x_1, \dots, x_n]$ with $q(x) \neq 0$ and $f(x) = p(x)/q(x)$ for all $x \in X_0$, cf. [2, p. 62]. Set

$$U := \{z \in X_{\mathbb{C}} \mid q(z) \neq 0\}$$

and define $g: U \rightarrow \mathbb{C}$ by $g(z) = p(z)/q(z)$ for $z \in U$. It suffices to prove that

$$(i) \quad G(f)_{\mathbb{C}} \cap (U \times \mathbb{C}) = \text{graph}(g).$$

To this end denote by G the Zariski closure of $\text{graph}(g)$ in $\mathbb{C}^n \times \mathbb{C}$. Then G is irreducible with

$$\dim G = \dim X_{\mathbb{C}} = \dim X = \dim G(f)_{\mathbb{C}}.$$

Since $\text{graph}(g)$ is Zariski closed in $U \times \mathbb{C}$, it follows that

$$(ii) \quad G \cap (U \times \mathbb{C}) = \text{graph}(g).$$

Clearly, $\text{graph}(f|_{X_0}) \subset \text{graph}(g)$, which implies that $G(f)_{\mathbb{C}} \subset G$. Consequently,

$$(iii) \quad G(f)_{\mathbb{C}} = G,$$

both $G(f)_{\mathbb{C}}$ and G being complex algebraic sets of the same dimension. Hence (i) follows by combining (ii) and (iii). \square

Next we study the Zariski closure of Nash manifolds. One readily checks that the Zariski closure of a connected Nash submanifold of \mathbb{R}^n is an irreducible algebraic set. It is convenient to introduce the following notion.

We say that two Nash submanifolds A and B of \mathbb{R}^n are *compatible* if for any nonempty open subsets (in the relative strong topology) $A' \subset A$, $B' \subset B$ there exist points $a \in A'$, $b \in B'$ and an irreducible algebraic curve $C \subset \mathbb{R}^n$ with the following properties: a is an accumulation point of $A \cap (C \setminus \{a\})$ and b is an accumulation point of $B \cap (C \setminus \{b\})$. In that case, both a and b belong to C .

Proposition 2.2. *If two connected Nash submanifolds A and B of \mathbb{R}^n , with $\dim A = \dim B$, are compatible, then $\overline{A}^{\mathcal{Z}} = \overline{B}^{\mathcal{Z}}$.*

Proof. Note that $\overline{A}^{\mathcal{Z}}$ and $\overline{B}^{\mathcal{Z}}$ are irreducible algebraic subsets of the same dimension. Suppose that A and B are compatible, but $\overline{A}^{\mathcal{Z}} \neq \overline{B}^{\mathcal{Z}}$. Then the sets $A' := A \setminus \overline{B}^{\mathcal{Z}}$ and $B' := B \setminus \overline{A}^{\mathcal{Z}}$ are nonempty and open in A and B , respectively. Let $a \in A'$, $b \in B'$, and C be as in the definition of compatible Nash submanifolds given above. Since a is an accumulation point of $A \cap (C \setminus \{a\})$, it follows that the intersection $A \cap C$ is an infinite set and hence the irreducibility of C implies the inclusion $C \subset \overline{A}^{\mathcal{Z}}$. Thus we get a contradiction since $a, b \in C$. \square

We will need an affine version of the classical theorem of Bertini. For the sake of completeness we include a full proof of it.

Theorem 2.3 (Bertini). *Let $\pi: \mathbb{C}^n \times \mathbb{C}^k \rightarrow \mathbb{C}^n$ be the canonical projection, $Y \subset \mathbb{C}^n \times \mathbb{C}^k$ an irreducible algebraic set, and X the Zariski closure of $\pi(Y)$. If $\dim Y = \dim X = d \geq 2$, then the set of affine $(n - d + 1)$ -planes in \mathbb{C}^n contains a Zariski open dense subset \mathcal{B} such that for every $L \in \mathcal{B}$ the intersection $Y \cap \pi^{-1}(L)$ is an irreducible curve.*

Proof. We will repeat almost word by word the proof of Theorem 3.3.1 (a projective version of Bertini's theorem) from the excellent survey of R. Lazarsfeld [12]. First we fix a general affine $(n - d)$ -plane Λ such that $\pi^{-1}(\Lambda)$ cuts Y transversally in finitely many smooth points. By a translation we may assume that Λ is actually a linear subspace of \mathbb{C}^n . The space of linear $(n - d + 1)$ -planes which contain Λ is parametrized by a projective space $T = \mathbb{P}^{d-1}$. Given $t \in T$ we denote by L_t the corresponding linear $(n - d + 1)$ -plane. Consider the set

$$V := \{(y, t) \in Y \times T \mid \pi(y) \in L_t\}.$$

The issue is to establish the irreducibility of a general fiber of the second projection

$$p: V \rightarrow T.$$

To this end note that the first projection $V \rightarrow Y$ is actually the blowing up of Y along the finite set $Y \cap \pi^{-1}(\Lambda)$. Hence V is irreducible. Furthermore, if we fix a point $y_0 \in Y \cap \pi^{-1}(\Lambda)$, then the mapping $s: T \rightarrow V$, defined by $s(t) = (y_0, t)$, defines a global section of p whose image is away from the singular locus of V . Moreover, p is a submersion at each point $s(t) = (y_0, t)$, $t \in T$.

Let $Z \subset V$ be the union of singular points of V and critical points of p . Clearly Z is nowhere dense in V , hence $V \setminus Z$ is connected and p is a submersion on $V \setminus Z$. By a result of Verdier [14, Corollary 5.1], there exists a Zariski closed nowhere dense set $R \subset T$ such that

$$p': V' \rightarrow T'$$

is a locally trivial fibration (for the strong topology), where $T' = T \setminus R$, $V' = p^{-1}(T')$, and p' stands for the restriction of p to V' . Clearly s restricted to T' is a section of p' .

We claim that for each $t \in T'$ the fiber $p'^{-1}(t)$ is connected, which implies that $p^{-1}(t)$ is irreducible. Indeed, let W_t be the connected component of $p'^{-1}(t)$ containing $s(t)$ and set

$$W := \bigcup_{t \in T'} W_t.$$

The fiber $p'^{-1}(t)$ has finitely many connected components, and hence W_t is open and closed in it. Moreover, since p' is a locally trivial fibration, it follows that W is open and closed in V' . Hence $W = V'$ and $W_t = p'^{-1}(t)$. Thus $p'^{-1}(t)$ is connected. \square

We will make use of Theorem 2.3 to study real algebraic sets. First we recall that an algebraic subset V of \mathbb{C}^n is said to be *defined over \mathbb{R}* if it is the set of common zeros (in \mathbb{C}^n) of a collection of polynomials with real coefficients. In other words, V is required to be stable under complex conjugation. In that case,

$$V(\mathbb{R}) := V \cap \mathbb{R}^n$$

is called the set of real points of V . If $V(\mathbb{R})$ is Zariski dense in V , then

$$V(\mathbb{R})_{\mathbb{C}} = V.$$

Proposition 2.4. *Let X be a d -dimensional irreducible algebraic subset of \mathbb{R}^n . Let A and B be Nash submanifolds of \mathbb{R}^n , both of dimension d and contained in X . Let $A' \subset A$ and $B' \subset B$ be nonempty open subsets (in the relative strong topology). Then there exist points $a \in A'$, $b \in B'$ and an affine $(n - d + 1)$ -plane M in \mathbb{R}^n such that $C := X \cap M$ is an irreducible curve for which a is an accumulation point of $A \cap (C \setminus \{a\})$ and b is an accumulation point of $B \cap (C \setminus \{b\})$. In particular, A and B are compatible.*

Proof. It suffices to consider the case $d \geq 2$. Set $r := n - d + 1$. If X_0 is the set of regular points of X , then the subsets $A' \cap X_0 \subset A$ and $B' \cap X_0 \subset B$ are nonempty and open. By Theorem 2.3 (with $k = 0$), for a general affine r -plane L in \mathbb{C}^n the intersection $X_{\mathbb{C}} \cap L$ is an irreducible complex curve. We can choose such an L so that it is defined over \mathbb{R} and the affine r -plane $M := L(\mathbb{R})$ in \mathbb{R}^n cuts A and B transversally at some points $a \in A' \cap X_0$ and $b \in B' \cap X_0$. Then the complex curve $X_{\mathbb{C}} \cap L$ is defined over \mathbb{R} , and hence

$$C := (X_{\mathbb{C}} \cap L)(\mathbb{R}) = X \cap M$$

is an irreducible real curve satisfying the required conditions. \square

3. PROOF OF THEOREM A AND RELATED RESULTS

The following result will play the key role.

Theorem 3.1. *Let X be an irreducible algebraic subset of \mathbb{R}^n of dimension $d \geq 2$, and $f: X \rightarrow \mathbb{R}$ a semialgebraic function. Let D be a Nash submanifold of \mathbb{R}^n , contained in X and of dimension d . Let \mathcal{C} be the collection of all irreducible algebraic curves in X of the form $X \cap M$ for some affine $(n - d + 1)$ -plane M in \mathbb{R}^n with $D \cap M \neq \emptyset$. Assume that for every curve $C \in \mathcal{C}$ the restriction $f|_C$ is a rational function. Then the function f is rational.*

Proof. Let A and B be connected Nash submanifolds of \mathbb{R}^n , both of dimension d with $A \subset D$ and $B \subset X$, for which the restrictions $f|_A$ and $f|_B$ are analytic functions. Such A and B exist, the function f being semialgebraic, cf. [2].

Claim 1. The graphs $F := \text{graph}(f|_A)$ and $G := \text{graph}(f|_B)$ have the same Zariski closure in $\mathbb{R}^n \times \mathbb{R}$.

Note that F and G are connected Nash submanifolds of $\mathbb{R}^n \times \mathbb{R}$ of dimension d . By Proposition 2.2, it suffices to show that F and G are compatible. To this end let A' and B' be nonempty open subsets of A and B , respectively. According to Proposition 2.4, there exist points $a \in A'$, $b \in B'$ and a curve $C \in \mathcal{C}$ such that a is an accumulation point of $A \cap (C \setminus \{a\})$ and b is an accumulation point of $B \cap (C \setminus \{b\})$. Since $f|_C$ is a rational function, there exists a set $C_0 \subset C$ such that its complement $C \setminus C_0$ is finite, $f|_C$ is regular on C_0 , and

$$G(f|_C) = \overline{\text{graph}(f|_{C_0})}^{\mathcal{Z}}$$

is an irreducible algebraic curve in $\mathbb{R}^n \times \mathbb{R}$. The points a and b may not be in C_0 , but it does not matter. By construction, $(a, f(a))$ and $(b, f(b))$ are accumulation points of

$$F \cap (\text{graph}(f|_{C_0}) \setminus \{(a, f(a))\}) \text{ and } G \cap (\text{graph}(f|_{C_0}) \setminus \{(b, f(b))\}),$$

respectively. This argument shows that F and G are compatible, which completes the proof of Claim 1.

Note that the algebraic subset

$$Y := \overline{F}^{\mathcal{Z}}$$

of $X \times \mathbb{R}$ is of dimension d and irreducible.

Claim 2. There exists a nonempty Zariski open set $X_0 \subset X$ such that $\text{graph}(f|_{X_0}) \subset Y$.

Since the function f is semialgebraic, there is a decomposition

$$X = E \cup A_1 \cup \dots \cup A_k$$

into disjoint semialgebraic sets such that $\dim E < d$, and for $i = 1, \dots, k$ the A_i is a d -dimensional connected Nash submanifold of \mathbb{R}^n for which the restriction $f|_{A_i}$

is an analytic function, cf. [2]. By Claim 1, the Zariski closure of $\text{graph}(f|_{A_i})$ is equal to Y . Hence Claim 2 holds for $X_0 := X \setminus \overline{E}^{\mathbb{Z}}$.

Let $\pi: \mathbb{C}^n \times \mathbb{C} \rightarrow \mathbb{C}^n$ be the canonical projection. Obviously, the complexifications $Y_{\mathbb{C}} \subset \mathbb{C}^n \times \mathbb{C}$ and $X_{\mathbb{C}} \subset \mathbb{C}^n$ are irreducible complex algebraic sets of dimension d . We have $\pi(Y_{\mathbb{C}}) \subset X_{\mathbb{C}}$ since $\pi(Y) \subset X$ and π is continuous in the Zariski topology. It follows that the restriction $\pi_0: Y_{\mathbb{C}} \rightarrow X_{\mathbb{C}}$ of π is generically finite-to-one. Hence there exist a positive integer l and a nonempty Zariski open set $U \subset X_{\mathbb{C}}$ such that for every point $x \in U$ the set $\pi_0^{-1}(x) = Y_{\mathbb{C}} \cap \pi^{-1}(x)$ consists of l distinct points.

Claim 3. The map $\pi_0: Y_{\mathbb{C}} \rightarrow X_{\mathbb{C}}$ is generically one-to-one, that is, $l = 1$.

Set $r := n - d + 1$. By Theorem 2.3 (with $k = 0$), the set of affine r -planes in \mathbb{C}^n contains a Zariski open and dense subset \mathcal{B} such that for every $L \in \mathcal{B}$ the intersection $X_{\mathbb{C}} \cap L$ is an irreducible complex curve. Shrinking \mathcal{B} if necessary and making use of Theorem 2.3 (with $k = 1$), we may assume that $Y_{\mathbb{C}} \cap \pi^{-1}(L)$ is also an irreducible complex curve. Now we choose an affine r -plane M in \mathbb{R}^n such that $M_{\mathbb{C}} \in \mathcal{B}$ and M cuts X transversally at some regular point contained in $A \cap X_0 \cap U$. Then $X_{\mathbb{C}} \cap M_{\mathbb{C}}$ is an irreducible complex curve defined over \mathbb{R} , and

$$\Gamma := X \cap M$$

is its set of real points. By construction, Γ is an irreducible real curve with

$$\Gamma_{\mathbb{C}} = X_{\mathbb{C}} \cap M_{\mathbb{C}}.$$

Since the restriction $f|_{\Gamma}$ is a rational function, there exists a set $\Gamma_0 \subset \Gamma$ such that its complement $\Gamma \setminus \Gamma_0$ is finite, $f|_{\Gamma}$ is regular on Γ_0 , and

$$G(f|_{\Gamma}) = \overline{\text{graph}(f|_{\Gamma_0})}^{\mathbb{Z}}$$

is an irreducible algebraic curve in $\mathbb{R}^n \times \mathbb{R}$. By Claim 2,

$$\text{graph}(f|_{\Gamma_0 \cap X_0}) \subset Y \cap \pi^{-1}(M).$$

Since the intersection $\Gamma_0 \cap X_0$ is nonempty, we obtain $G(f|_{\Gamma}) \subset Y \cap \pi^{-1}(M)$. Taking the complexifications we get $G(f|_{\Gamma})_{\mathbb{C}} \subset Y_{\mathbb{C}} \cap \pi^{-1}(M_{\mathbb{C}})$. The last inclusion implies that

$$G(f|_{\Gamma})_{\mathbb{C}} = Y_{\mathbb{C}} \cap \pi^{-1}(M_{\mathbb{C}})$$

since both $G(f|_{\Gamma})_{\mathbb{C}}$ and $Y_{\mathbb{C}} \cap \pi^{-1}(M_{\mathbb{C}})$ are irreducible complex curves. According to Lemma 2.1, the equality

$$G(f|_{\Gamma})_{\mathbb{C}} \cap \pi^{-1}(x) = \{(x, f(x))\}$$

holds for all $x \in \Gamma_0$. It follows that

$$\pi_0^{-1}(x) = Y_{\mathbb{C}} \cap \pi^{-1}(x) = \{(x, f(x))\}$$

for all $x \in \Gamma_0$. Since the intersection $\Gamma_0 \cap U$ is nonempty, we conclude that $l = 1$, which proves Claim 3.

We are ready to complete the proof of the theorem. Obviously, $Y_{\mathbb{C}} \cap \pi^{-1}(U)$ is a constructible set, which according to Claim 3 is the graph of some function

$g: U \rightarrow \mathbb{C}$. By Zariski's theorem on constructible graph (see for example Lojasiewicz [13, p. 449]), there exist a nonempty Zariski open set $U' \subset U$ and complex polynomial functions p, q on \mathbb{C}^n such that

$$q(z) \neq 0 \text{ and } g(z) = \frac{p(z)}{q(z)} \text{ for all } z \in U'.$$

It follows from Claim 2 that $g(x) = f(x) \in \mathbb{R}$ for all $x \in X_0 \cap U'$. In particular, using the standard notation for complex conjugation, we get

$$(*) \quad f(x) = \frac{p(x)}{q(x)} = \frac{\overline{p(x)}}{\overline{q(x)}} \text{ for all } x \in X_0 \cap U'.$$

The polynomials

$$p_1(z) := p(z)\overline{q(\bar{z})} + \overline{p(\bar{z})}q(z) \text{ and } q_1(z) := 2q(z)\overline{q(\bar{z})}$$

satisfy $\overline{p_1(\bar{z})} = p_1(z)$ and $\overline{q_1(\bar{z})} = q_1(z)$, which implies that they have real coefficients. In view of (*) we get

$$f(x) = \frac{p_1(x)}{q_1(x)} \text{ for all } x \in X_0 \cap U'.$$

Hence f is a rational function on X , as required. \square

As an immediate consequence of Theorem 3.1 we obtain the following criterion for rationality of semialgebraic functions on \mathbb{R}^n .

Corollary 3.2. *Let U be a nonempty open subset (in the strong topology) of \mathbb{R}^n . A semialgebraic function on \mathbb{R}^n is rational, provided that its restriction to every affine line passing through a point in U is a rational function.*

Let us note that the hypothesis in Corollary 3.2 cannot be relaxed too much.

Example 3.3. The function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$, defined by $f(x, y) = (x^4 + y^4)^{\frac{1}{2}}$, is semialgebraic and arc-analytic. The restriction of f to an affine line $L \subset \mathbb{R}^2$ is a rational function if and only if L passes through the origin. Obviously, f is not a rational function.

It is convenient to record the following observation (cf. [5, p. 91]).

Remark 3.4. Let X be an algebraic subset of \mathbb{R}^n , and $f: X \rightarrow \mathbb{R}$ a function. Then f is hereditarily rational if and only if there exists a sequence of algebraic sets

$$X = X_0 \supset X_1 \supset \cdots \supset X_m = \emptyset$$

such that for $i = 0, \dots, m-1$ the restriction $f|_{X_i}$ is regular on $X_i \setminus X_{i+1}$. In particular, every hereditarily rational function on X is semialgebraic. Indeed, set $X_0 := X$. If f is rational, then there exists an algebraic subset $X_1 \subset X_0$ such that $\dim X_1 < \dim X_0$ and f is regular on $X_0 \setminus X_1$. If f is hereditarily rational, we can repeat this process with $f|_{X_1}$, and so on, which yields a sequence of algebraic sets with the required properties and shows that f is semialgebraic. The converse readily follows.

Next we deal with the extension problem for hereditarily rational functions.

Proposition 3.5. *Let $W \subset X$ be algebraic subsets of \mathbb{R}^n , and $f: X \rightarrow \mathbb{R}$ a hereditarily rational function that is regular on $X \setminus W$. Then f can be extended to a hereditarily rational function on \mathbb{R}^n that is regular on $\mathbb{R}^n \setminus W$.*

Proof. We use induction on $\dim X$. The case $\dim X$ is obvious.

Since f is a rational function, it is regular on a Zariski open dense set $X_0 \subset X$. We may assume that $W \subset X \setminus X_0$ so, in particular, $\dim W < \dim X$. Hence, by induction, $f|_W$ can be extended to a hereditarily rational function $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}$ that is regular on $\mathbb{R}^n \setminus W$. The function $g := f - \varphi|_X$ on X is hereditarily rational, regular on $X \setminus W$ and vanishes on W . It suffices to extend g to a hereditarily rational function $G: \mathbb{R}^n \rightarrow \mathbb{R}$ that is regular on $\mathbb{R}^n \setminus W$. This can be done as follows. Since g is regular on $X \setminus W$, we can find polynomial functions p, q on \mathbb{R}^n satisfying

$$q(x) \neq 0 \text{ and } g(x) = \frac{p(x)}{q(x)} \text{ for all } x \in X \setminus W,$$

cf. [2, p. 62]. Let r be a polynomial function on \mathbb{R}^n vanishing precisely in X . Set $P := pq$, $Q := q^2 + r^2$, and define $G: \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$G(x) = \frac{P(x)}{Q(x)} \text{ for } x \in \mathbb{R}^n \setminus W \text{ and } G(x) = 0 \text{ for } x \in W.$$

By construction, G is regular on $\mathbb{R}^n \setminus W$ and $G|_X = g$. Furthermore, G is hereditarily rational in view of Remark 3.4. The proof is complete. \square

Now we can prove the main result of this paper.

Proof of Theorem A. It is clear that (b) \Rightarrow (a), whereas (a) \Rightarrow (b) follows from Proposition 3.5. By Remark 3.4, (a) \Rightarrow (c). Thus it remains to show that (c) \Rightarrow (a).

Suppose that condition (c) holds. We want to prove that for every algebraic set $Z \subset X$ the restriction $f|_Z$ is a rational function. It suffices to do it for Z irreducible of dimension at least 2. In that case, however, the assertion follows from Theorem 3.1. \square

4. NASH ARCS AND MEROMORPHIC FUNCTIONS

Throughout this section, X denotes an algebraic subset of \mathbb{R}^n . A map $\gamma: (-1, 1) \rightarrow X$ that is analytic and semialgebraic is called a *Nash arc* in X .

Lemma 4.1. *Let $f: X \rightarrow \mathbb{R}$ be a continuously curve-rational function, and $\gamma: (-1, 1) \rightarrow X$ a Nash arc. Then the function $f \circ \gamma$ is analytic.*

Proof. We may assume that γ is a nonconstant map. Then $\gamma((-1, 1))$ is a semi-algebraic curve, and hence its Zariski closure C is an irreducible algebraic curve in X . By assumption, $f|_C$ is a continuous rational function. In particular, there

exist two polynomials $p, q \in \mathbb{R}[x_1, \dots, x_n]$ with $q(x) \neq 0$ and $f(x) = p(x)/q(x)$ for all $x \in C \setminus C_1$, where C_1 is a finite set. The function

$$f(\gamma(t)) = (f|_C)(\gamma(t)) = \frac{p(\gamma(t))}{q(\gamma(t))}$$

is continuous and meromorphic on $(-1, 1)$, and hence it is analytic. \square

Lemma 4.2. *Let $f: X \rightarrow \mathbb{R}$ be a semialgebraic function. Assume that for every Nash arc $\gamma: (-1, 1) \rightarrow X$ the function $f \circ \gamma$ is analytic. Then f is continuous.*

Proof. Let $\mathbb{S}^1 \subset \mathbb{R}^2$ be the unit circle. We compactify \mathbb{R} via the embedding $\mathbb{R} \rightarrow \mathbb{S}^1$, which is the inverse of the stereographic projection from the north pole, and regard $F := \text{graph}(f)$ as a subset of $\mathbb{R}^n \times \mathbb{R}^2$. Fix a point $x_0 \in X$ and let $l \in \mathbb{S}^1$ be any point such that (x_0, l) belongs to the closure of F in $\mathbb{R}^n \times \mathbb{R}^2$. It suffices to prove that $f(x_0) = l$. By the Nash curve selection lemma [2, Proposition 8.1.13], there exists a Nash arc $\varphi = (\psi, g): (-1, 1) \rightarrow \mathbb{R}^n \times \mathbb{R}^2$ with $\varphi(0) = (\psi(0), g(0)) = (x_0, l)$ and $\varphi((0, 1)) \subset F$. In particular, $g(t) = f(\psi(t))$ for $t \in (0, 1)$. Consequently, $g(t) = f(\psi(t))$ for $t \in (-1, 1)$ since both g and $f \circ \psi$ are analytic functions. Hence

$$l = g(0) = f(x_0) = \lim_{t \rightarrow 0} f(\psi(t)),$$

which proves the continuity of f at x_0 . \square

Let M be a connected real analytic manifold. We say that a function $\lambda: M \rightarrow \mathbb{R}$ is *meromorphic* if there exist two analytic functions $\alpha: M \rightarrow \mathbb{R}$ and $\beta: M \rightarrow \mathbb{R}$ such that the set $M_0 := \{y \in M \mid \beta(y) \neq 0\}$ is nonempty and $\lambda(y) = \alpha(y)/\beta(y)$ for all $y \in M_0$.

Proposition 4.3. *Let $f: X \rightarrow \mathbb{R}$ be a hereditarily rational function. Then for any connected real analytic manifold M and any analytic map $\varphi: M \rightarrow X$ the function $f \circ \varphi$ is meromorphic.*

Proof. We first note that the Zariski closure Z of $\varphi(M)$ is an irreducible algebraic subset of X (possibly with $\dim Z > \dim M$). By assumption, the restriction $f|_Z$ is a rational function, and hence there exist two polynomials $p, q \in \mathbb{R}[x_1, \dots, x_n]$ with $q(x) \neq 0$ and $f(x) = p(x)/q(x)$ for all $x \in Z \setminus Z_1$, where $Z_1 \subset Z$ is an algebraic set, $Z_1 \neq Z$. Consequently, $f \circ \varphi$ is a meromorphic function since the set $M_0 := \varphi^{-1}(Z \setminus Z_1)$ is nonempty and $f(\varphi(y)) = p(\varphi(y))/q(\varphi(y))$ for all $y \in M_0$. \square

Proof of Theorem B. Let X be an algebraic subset of \mathbb{R}^n , and $f: X \rightarrow \mathbb{R}$ a semialgebraic function that is continuously curve-rational. According to Theorem A, f is hereditarily rational. Furthermore, by Lemmas 4.1 and 4.2, f is continuous. Now, let $\eta: (-1, 1) \rightarrow X$ be an analytic arc. In view of Proposition 4.3, $f \circ \eta$ is a meromorphic function. Thus, $f \circ \eta$ is analytic since it is continuous and meromorphic on $(-1, 1)$. Consequently, f is arc-analytic. \square

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